NASA TECHNICAL REPORT

NASA TR R-341

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NASA TR R-341

INTEGRATION OF THE RELATIVISTIC EQUATIONS OF MOTION OF AN ARTIFICIAL EARTH SATELLITE

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JULY 1970

1. Report No. NASA TR R-341	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle Integration of the Relativistic Equations of Motion of an Artificial Earth Satellite		5. Report Date July 1970
		6. Performing Organization Code
7. Author(s) A. Ghaffari		8. Performing Organization Report No. G-978
9. Performing Organization Name and Address Goddard Space Flight Center Greenbelt, Maryland 20771		10. Work Unit No. 311-07-21-01-51
		11. Contract or Grant No.
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		13. Type of Report and Period Covered Technical Report
		14. Sponsoring Agency Code
15. Supplementary Notes		

16. Abstract

The Lindstedt perturbation method is applied to the motion of an artificial Earth satellite that moves along a geodesic of the Schwarzschild metric of general relativity. The purpose of this analysis is to determine the extent to which general-relativistic effects are detectable in range measurements of Earth-orbiting spacecraft.

17. Key Words Suggested by Author General relativity Lindstedt method Satellite orbits Celestial mechanics	18. Distribution Statement Unclassified–Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price *
Unclassified	Unlimited	9	$3.00

*For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151

SUMMARY

The Lindstedt perturbation method is applied to the motion of an artificial Earth satellite that moves along a geodesic of the Schwarzschild metric of general relativity theory. It is shown that both the amplitude and the frequency of the first-order approximate solution obtained are affected by the nonlinearity of the relativistic term appearing in the equation of motion. The approximate periodic solution is compared with the solution for the motion of a synchronous artificial Earth satellite (ATS 3) in a Newtonian force field. Under the assumption that the coordinate time and the initial conditions are the same in both systems, it is deduced that the maximum of the magnitude difference between the two calculated radii is of order 1.6 cm after one-half orbit and is, therefore, too small to be detected.

CONTENTS

INTEGRATION OF THE RELATIVISTIC EQUATIONS OF MOTION OF AN ARTIFICIAL EARTH SATELLITE

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INTRODUCTION

In the theory of general relativity, the external gravitational field of a spherically symmetric massive body M whose center lies at $r = 0$ is represented by the static Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{1}{c^2}\left[\left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad . \tag{1}$$

In this metric, an exact solution for a set of discrete bodies is possible only when one of the bodies is of finite mass and all of the rest are of infinitesimally small mass (Reference 1). If the massive body is taken to be the Earth, then an artificial Earth satellite may be considered as an infinitesimal test particle whose gravitational field may be neglected.

The complete solution for the relativistic effects of the combined fields of the Earth and the Sun acting on an artificial Earth satellite is complex. The metric must involve both the principal masses, and since the Earth is orbiting the Sun, the field is no longer static. It is well known (Reference 2) that, in general, the motion of an Earth satellite can be derived with a great deal of precision through consideration of the effects of the Earth and the Sun separately. The relativistic effects of solar gravitation have been calculated by Papapetrou (Reference 3) and Corinaldesi and Papapetrou (Reference 4). The use of artificial Earth satellites seems more suitable than the use of solar satellites for testing the effects of general relativity because the Sun has many planets and their mutual attractions are significant.

For convenience, it is assumed that the Earth's gravitational field is spherically symmetric and that the Sun's gravitational effect upon an artificial Earth satellite is negligible. With these assumptions, one knows that the motion of an artificial Earth satellite is given by the equations of the ordinary geodesics of the Schwarzschild metric (References 1 and 5) which combined into one simplified form is

*Some of the results of this paper have been presented at the Sixth Semiannual Astrodynamic Conference held at Goddard Space Flight Center, Greenbelt, Maryland, on November 7 and 8, 1967.

$$\left(\frac{d(1/r)}{d\phi}\right)^2 = 2\frac{m}{r^3} - \frac{1}{r^2} + 2m\frac{c^2}{h^2 r} - \gamma ,$$

or

$$\left(\frac{du}{d\phi}\right)^2 = 2mu^3 - u^2 + \frac{2mc^2}{h^2} u - \gamma , \tag{2}$$

where

$$u = \frac{1}{r} ,$$

$$\gamma = \frac{c^2}{h^2}(1 - \beta^2) ,$$

$$\beta = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} ,$$

$$m = \frac{GM}{c^2} ,$$

and

$$h = r^2 \frac{d\phi}{ds}$$

and

r = distance of the satellite from the center of the Earth,

M = mass of the Earth,

v = speed of the satellite,

c = local speed of light,

h, β = constants of integration,

ϕ = true anomaly,

and G = gravitational constant.

INTEGRATION OF THE EQUATIONS OF MOTION

The classical equation [equation (2)] has been investigated and discussed completely in most standard texts on celestial mechanics and general relativity (References 5, 6, 7, 8, and 9). The rigorous integration of equation (2) leads to the Weirstrass elliptic function σ (ϕ, g_2, g_3), which satisfies the differential equation

$$\left(\frac{d\sigma}{d\phi}\right)^2 = 4\sigma^3 - g_2\sigma - g_3 ,$$

where g_2 and g_3 are constants. However, in practice, an approximate integration of equation (2) gives the advance of the perihelion. The solution of equation (2), with the Sun as the central body and a planet as the test particle, gave the perihelion advances in 100 years for the planets Mercury (43".03), Venus (8".64), Earth (3".84), and Mars (1".35) predicted by Einstein's law of gravitation. A complete discussion of equation

(2) has been given by Chazy (Reference 6), and exhaustive research of the orbits defined by equation (2) with the location of their singularities has been performed by Hagihara (Reference 10). McVittie (Reference 5) discussed and solved equation (2) by a different method.

Differentiating equation (2) with respect to the true anomaly ϕ and setting $mc^2/h^2 = 1/p$, gives

$$\frac{d^2u}{d\phi^2} + u = \frac{1}{p} + 3mu^2, \tag{3}$$

provided $du/d\phi$ has only isolated zeros (i.e., eliminating the circular-orbit solutions). In the Newtonian law of gravitation, the right-hand sides of equations (2) and (3) are quadratic and linear in u, respectively. The presence of corrective terms $2mu^3$ in equation (2) and $3mu^2$ in equation (3) arises from Einstein's law of gravitation.

Equation (3) was first derived by Eddington (Reference 7). Using a method of successive approximation, he obtained a Keplerian orbit as a first approximation by ignoring the small term $3mu^2$. Then, substituting the first approximation into the small term $3mu^2$, he arrived at a second approximation with a secular term describing the resonant case. Bergmann (Reference 9) also considered equation (3) and, applying a Fourier series procedure, he obtained the perihelion advances of the planets without giving the explicit expressions for the planets' orbits.

In the sections that follow,

(1) Equation (3) is integrated by the classical Lindstedt method to give a first approximate periodic solution starting from the perigee.
(2) The approximate periodic solution is compared with the solution obtained for the motion of an artificial Earth satellite in a Newtonian force field.
(3) A numerical estimate of the comparison is made with the assumption that the central body is the Earth and the test particle is a long-life, synchronous Earth satellite such as the Application Technology Satellite 3 (ATS 3).

The advantages of equation (3) over equation (2) are twofold:

(1) Though equation (3) is of second order, it is, nevertheless, nonlinear in u only, whereas equation (2) is nonlinear in both u and $du/d\phi$, and its integration leads to the Weierstrass elliptic functions.
(2) We are primarily concerned with the periodic solution and its behavior of periodicity. The left-hand side of equation (3) clearly suggests the application of classical and modern approximation methods for the derivation of approximations to periodic solutions.

APPLICATION OF THE LINDSTEDT METHOD

Let us assume that we wish to find an approximate periodic solution of equation (3) satisfying the following initial conditions (i.e., starting from the perigee):

$$u(0) = \frac{1 + e}{p} \quad \text{and} \quad \left(\frac{du}{d\phi}\right)_0 = 0. \tag{4}$$

In order to find such a solution and to investigate the perturbation in the basic frequency arising from the presence of the relativistic term on the right-hand side of equation (3), and also to eliminate the secular term, the Lindstedt method (Reference 11) is applied to equation (3).

The Lindstedt method consists primarily of a change of the independent variable from ϕ to another independent variable α such that the determination of the available unknown coefficients enables us to eliminate gradually the secular terms in the subsequent approximations (Reference 12). As the frequency is altered, it is of advantage to replace the independent variable ϕ with a new independent variable α in a manner defined by the relation

$$\phi = \alpha(1 + c_1\epsilon + c_2\epsilon^2 + \ldots) , \tag{5}$$

where c_1, c_2, ... are unknown coefficients and ϵ is an arbitrarily small positive parameter. The smallness of the gravitational radius m enables us to suppose that $\epsilon = 3m$. Then, equation (3) becomes

$$\frac{d^2u}{d\alpha^2} + \left(1 + c_1\epsilon + c_2\epsilon^2 + \ldots\right)^2 \left(u - \frac{1}{p} - \epsilon u^2\right) = 0 ,$$

or

$$\frac{d^2u}{d\alpha^2} + u - \frac{1}{p} + \epsilon\left(-\frac{2c_1}{p} + 2c_1 u - u^2\right) + o(\epsilon^2) = 0 . \tag{6}$$

The solution of equation (6) can be written as a power series with respect to the small parameter ϵ:

$$u(\epsilon, \alpha) = \sum_{n=0}^{\infty} \epsilon^n u_n(\alpha) . \tag{7}$$

To limit ourselves to the first-order approximation, the series is written

$$u = u_0 + \epsilon u_1 + o(\epsilon^2) . \tag{8}$$

Using equations (6) and (7), one finds that the leading term u_0 is a solution of the unperturbed equation

$$\frac{d^2u_0}{d\alpha^2} + u_0 = \frac{1}{p} \tag{9}$$

and that u_1 satisfies

$$\frac{d^2u_1}{d\alpha^2} + u_1 = \frac{2 + e^2}{2p^2} - \frac{2c_1}{p} + \frac{2e}{p^2}(1 - c_1 p) \cos\alpha + \frac{e^2}{2p^2} \cos 2\alpha . \tag{10}$$

The unknown coefficient c_1 is to be determined such that no secular term appears in the solution of equation (10). Hence, we choose c_1 so that

$$c_1 = \frac{1}{p}, \tag{11}$$

and, therefore, equation (10) assumes the form

$$\frac{d^2u_1}{d\alpha^2} + u_1 = \frac{1}{2p^2}(e^2 - 2 + e^2 \cos 2\alpha). \tag{12}$$

Equation (12) has the solution

$$u_1 = \frac{e^2 - 2}{2p^2}(1 - \cos 2\alpha) \tag{13}$$

satisfying the initial conditions:

$$\left.\begin{aligned} u_1(0) &= 0 , \\ \left(\frac{du_1}{d\alpha}\right)_0 &= 0 . \end{aligned}\right\} \tag{14}$$

Therefore, the first-order approximate periodic solution of equation (3) satisfying the initial conditions [equations (4)] is

$$u = u_0 + \epsilon u_1 + o(\epsilon^2) = \frac{1 + e \cos \alpha}{p} + \frac{3m(e^2 - 2)}{2p^2}(1 - \cos 2\alpha) + o(\epsilon^2) , \tag{15}$$

where

$$\alpha = \frac{\phi}{1 + c_1\epsilon + \ldots} = \frac{\phi}{1 + c_1\epsilon} + o(\epsilon^2) = \left(1 - \frac{3m}{p}\right)\phi + o(\epsilon^2) . \tag{16}$$

We can deduce that the nonlinearity shown by the relativistic term in Equation 3 affects not only the amplitude but also the frequency of the solution.

Equations (15) and (16) show that the change in the frequency depends upon the amplitude a and the eccentricity e of the Keplerian orbit and also upon the parameter $\epsilon = 3m$, a property of the periodic solutions of all nonlinear autonomous differential equations. The period of the approximate periodic solution of equation (9) is 2π, i.e., the orbits are closed. The period of the exact solution of equation (3) differs from 2π by a small angle δ which is the difference between the angular positions of two succeeding perigees and is given by

$$\delta = \frac{2\pi}{1 - (3m/p)} - 2\pi = 2\pi\left(1 + \frac{3m}{p}\right) - 2\pi = \frac{6m\pi}{p} = \frac{6m\pi}{a(1 - e^2)}\left(\frac{\text{rad}}{\text{rev}}\right). \tag{17}$$

Therefore, the precession of the perigee of a satellite orbit obtained by this method amounts to $6m\pi/p$ rad/rev, which is in close agreement with the precession predicted by Einstein's theory of general relativity as well as with the gravitational theories of Whitehead (Reference 13) and Birkhoff (Reference 14).

COMPARISON WITH NEWTONIAN FORCE FIELD AND NUMERICAL ANALYSIS

Now let a second artificial Earth satellite with the same characteristics as the first move in a planar elliptic orbit according to Newtonian laws of motion. If r_N and ϕ are the classical polar coordinates of the second satellite in the orbital plane and $u_N = 1/r_N$, the classical Newtonian equation of motion is

$$u_N = \frac{1 + e \cos \phi}{p}, \tag{18}$$

where the true anomaly ϕ is measured from the perigee and $p = a(1 - e^2)$ is the semi-latus rectum; a and e are the semi-major axis and the eccentricity of the Newtonian orbit, respectively. We assume that both satellites start from the perigee, i.e., they both satisfy the initial conditions:

$$\begin{aligned} (r_N)_0 &= (r_R)_0, \\ (u_N)_0 &= (u_R)_0, \\ (u'_N)_0 &= (u'_R)_0, \end{aligned} \tag{19}$$

where u_R and u_N stand for relativistic and Newtonian solutions, respectively.

For a comparison of the approximate periodic relativistic solution [equation (15)] with the classical Newtonian solution [equation (18)], the difference Δu is formed:

$$\Delta u = u_N - u_R = \frac{e}{p}\left[\cos \phi - \cos \left(1 - \frac{3m}{p}\right)\phi\right] - \frac{3m(e^2 - 2)}{2p^2}\left[1 - \cos 2\left(1 - \frac{3m}{p}\right)\phi\right]. \tag{20}$$

The difference Δu is a function of the Keplerian orbital elements and the gravitational radius m of the central body. Since the central body is taken to be the Earth, $m = 0.443$ cm, and the parameters a and e are constant for one revolution. Therefore, for a single revolution, the difference $\Delta u = \Delta u(\phi)$ is a function of the true anomaly ϕ only.

For a numerical estimate of $\Delta u(\phi)$, equation (20) was applied to orbital data of the ATS 3, which had the the elements

$$a = 6.6109161 \times R,$$

and

$$e = 0.0001703684,$$

where $R = 6.371 \times 10^8$ cm is the mean radius of the Earth. Setting

$$r_R = r_N + \sigma = r_N\left(1 + \frac{\sigma}{r_N}\right) \tag{21}$$

where σ denotes the difference between the relativistic and the Newtonian radii, one finds that

$$\Delta u = u_N - u_R = \frac{1}{r_N} - \frac{1}{r_R} = \frac{\sigma}{r_N^2}. \tag{22}$$

The results are listed in Table 1 and plotted in Figure 1 for the range $0 \leq \phi \leq 190°$. The function $\Delta u(\phi)$, being a continuous function of ϕ, begins to increase again after 180°. Table 1 and Figure 1 show that, for the case of the ATS 3, the relativistic correction to Equation 3 is too small to be detected, and the maximum value of $\Delta u \sim 1.5 \times 10^{-19}$ is obtained for $\phi = 89°59'59''.148792708$ (correct to the nine decimal places given in seconds). Therefore, the maximum radial deviation is of the order

$$\sigma \sim r_N^2 (1.5 \times 10^{-19}) = (4 \times 10^9 \text{ cm})^2 (1.5 \times 10^{-19}) \sim 1.6 \text{ cm}, \tag{23}$$

which is too small to be detected. However, it is important to separate the relativistic effects from those attributable to other causes, such as the oblateness of the Earth, magnetic fields, the influence of high-altitude winds, and so forth.

Table 1—Variation of $\Delta u(\phi)$ in terms of the true anomaly ϕ.

ϕ (deg)	$\Delta u(\phi)$
0	0
10	$.4517717214 \times 10^{-20}$
20	$.1752594324 \times 10^{-19}$
30	$.3745562899 \times 10^{-19}$
40	$.6190285723 \times 10^{-19}$
50	$.8791879840 \times 10^{-19}$
60	$.1123653936 \times 10^{-18}$
70	$.1322938634 \times 10^{-18}$
80	$.1453003889 \times 10^{-18}$
89 59'59''.148792708	$.149816062806 \times 10^{-18}$
90	$.1498160626 \times 10^{-18}$
100	$.1452961373 \times 10^{-18}$
110	$.1322857458 \times 10^{-18}$
120	$.1123541627 \times 10^{-18}$
130	$.8790553806 \times 10^{-19}$
140	$.6188892964 \times 10^{-19}$
150	$.3744260732 \times 10^{-19}$
160	$.1751553208 \times 10^{-19}$
170	$.4511663842 \times 10^{-20}$
180	$-.1050408293 \times 10^{-27}$
190	$.4525317707 \times 10^{-20}$

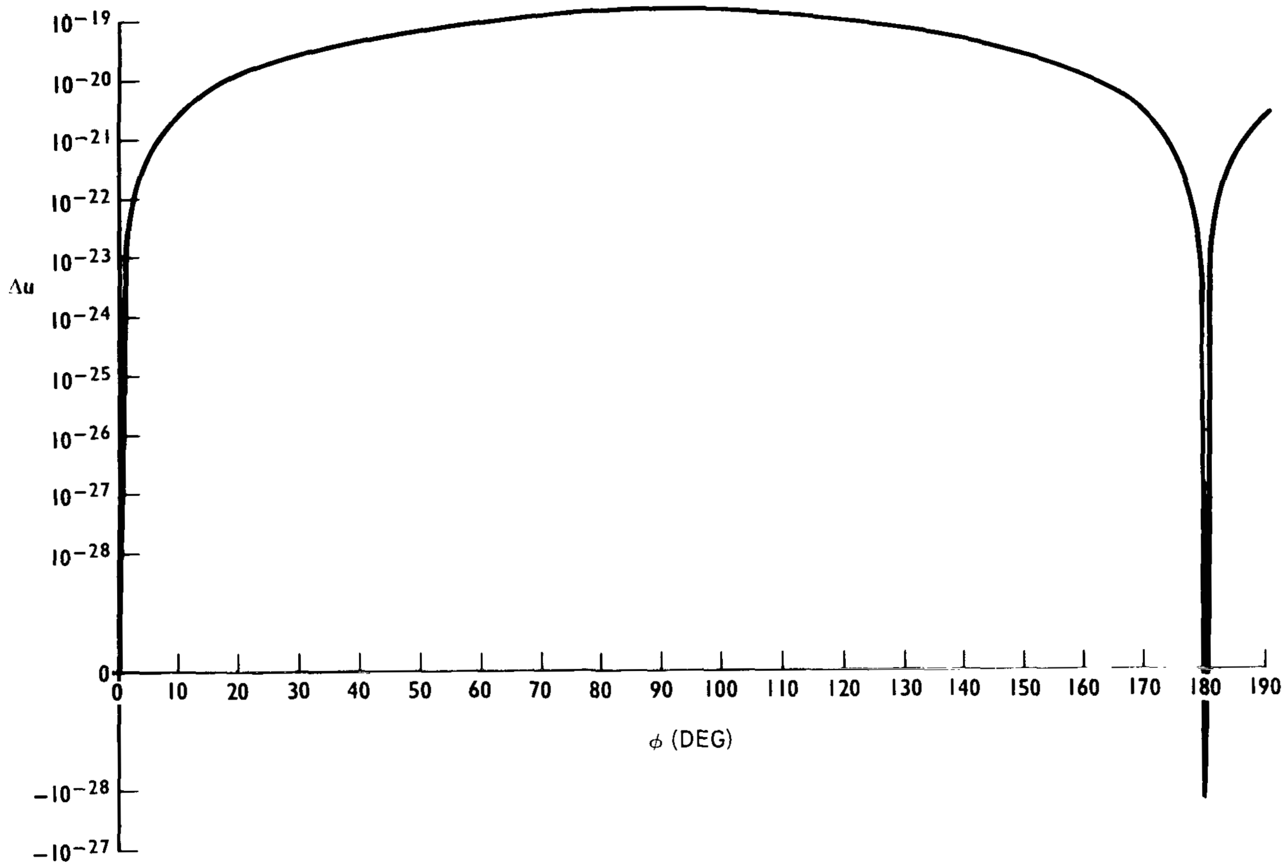

Figure 1—Radial component of range perturbation.

In a later paper we expect to report on the application of the averaging method of Bogoliubov and Mitropolsky (Reference 15) to equation (3), which leads to a different approximate solution, and its comparison with the solution obtained here [equation (15)]. A comparison of the obtained solution with that of Lass and Solloway (Reference 16) will also be included.

ACKNOWLEDGMENT

I would like to thank Mr. Paul E. Schmid, Jr., for reading the manuscript and for offering many useful suggestions.

Goddard Space Flight Center
National Aeronautics and Space Administration
Greenbelt, Maryland, March 12, 1970
311-07-21-01-51

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